

# Conformastationary disk-haloes in Einstein-Maxwell gravity

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An exact solution of the Einstein-Maxwell field equations for a conformastationary metric with magnetized disk-haloes sources is worked out in full. The characterization of the nature of the energy momentum tensor of the source is discussed. All the expressions are presented in terms of a solution of the Laplace's equation. A “generalization” of the Kuzmin solution of the Laplace's equations is used as a particular example. The solution obtained is asymptotically flat in general and turns out to be free of singularities. All the relevant quantities show a reasonable physical behaviour.

## I. INTRODUCTORY REMARKS

In a recent work [1], we presented a relativistic model describing a thin disk surrounded by a halo in presence of an electromagnetic field. The model was obtained by solving the Einstein-Maxwell equations on a particular conformastatic spacetime background and by using the distributional approach for the energy-momentum tensor. The class of solution corresponding to the model is asymptotically flat and singularity-free, and satisfies all the energy conditions. The purpose of the present work is to extend the above-mentioned study to the conformastationary case, and the Kuzmin solution of the Laplace's equation to include a “generalized” Kuzmin solution of the Laplace's equation. The reason to undertake such an endeavour are easy to understand. Indeed, the issue of the exact solution of the Einstein and Einstein-Maxwell equations describing isolated sources self gravitating in a stationary axially symmetric spacetime appears to be of great interest both from a mathematical and physical point of view. For details of the astrophysics importance and the most relevant developments concerning the disks and disk-haloes sources the reader is referred to the works [1, 2] and references therein.

In this work we present a new exact solution of the Einstein-Maxwell field equations for a thin disk surrounded by a magnetized halo in a conformastationary background. This solution is notoriously simple in its mathematical form. Moreover, the interpretation of the energy-momentum tensor presented here generalises the commonly used pressure free models to a fluid with non-vanishing pressure, heat flux and anisotropic tensor. In Section II we present an exact general relativistic model describing a disk surrounded by an electromagnetized halo and we obtain a solution of the Einstein-Maxwell field equations in terms of a solution of Laplace's equation. In Section III we express the surface energy-momentum tensor of the disk in the canonical form and we present a physical interpretation of it in terms of a fluid with non-vanishing pressure and heat flux. In Section IV a particular family of conformastationary magnetized disk-haloes solutions is presented. We complete the paper with a discussion of the results in Section V.

## II. GENERAL RELATIVISTIC MAGNETIZED HALOES SURROUNDING THIN DISKS

To obtain an exact general relativistic model describing a disk surrounded by an electromagnetized halo in a conformastationary spacetime, we solved the distributional Einstein-Maxwell field equations assuming axial symmetry and that the derivatives of the metric and electromagnetic potential across the disk space-like hyper-surface are discontinuous. To formulate the corresponding distributional form of the Einstein-Maxwell field equations, we introduce the usual cylindrical coordinates  $x^\alpha = (t, r, z, \varphi)$  and assume that there exists an infinitesimally thin disk located at the hypersurface  $z = 0$ , so that the metric and the electromagnetic potential can be written as

$$g_{\alpha\beta} = g_{\alpha\beta}^+ \theta(z) + g_{\alpha\beta}^- \{1 - \theta(z)\}, \quad (1a)$$

$$A_\alpha = A_\alpha^+ \theta(z) + A_\alpha^- \{1 - \theta(z)\}, \quad (1b)$$

respectively. Accordingly, the Ricci tensor reads

$$R_{\alpha\beta} = R_{\alpha\beta}^+ \theta(z) + R_{\alpha\beta}^- \{1 - \theta(z)\} + H_{\alpha\beta} \delta(z), \quad (2)$$

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where  $\theta(z)$  and  $\delta(z)$  are, respectively, the Heaviside and Dirac distributions with support on  $z = 0$ . Here  $g_{\alpha\beta}^{\pm}$  and  $R_{\alpha\beta}^{\pm}$  are the metric tensors and the Ricci tensors of the  $z \geq 0$  and  $z \leq 0$  regions, respectively, and

$$H_{\alpha\beta} = \frac{1}{2} \{ \gamma_{\alpha}^z \delta_{\beta}^z + \gamma_{\beta}^z \delta_{\alpha}^z - \gamma_{\mu}^{\mu} \delta_{\alpha}^z \delta_{\beta}^z - g^{zz} \gamma_{\alpha\beta} \}, \quad (3)$$

with  $\gamma_{\alpha\beta} = 2g_{\alpha\beta,z}$  and all the quantities are evaluated at  $z = 0^+$ . In agreement with (2) the energy-momentum tensor and the electric current density are expressed as

$$T_{\alpha\beta} = T_{\alpha\beta}^{+} \theta(z) + T_{\alpha\beta}^{-} \{1 - \theta(z)\} + Q_{\alpha\beta} \delta(z), \quad (4a)$$

$$J_{\alpha} = J_{\alpha}^{+} \theta(z) + J_{\alpha}^{-} \{1 - \theta(z)\} + \mathcal{I}_{\alpha} \delta(z), \quad (4b)$$

where  $T_{\alpha\beta}^{\pm}$  and  $J_{\alpha}^{\pm}$  are the energy-momentum tensors and electric current density of the  $z \geq 0$  and  $z \leq 0$  regions, respectively. Moreover,  $Q_{\alpha\beta}$  and  $\mathcal{I}_{\alpha}$  represent the part of the energy-momentum tensor and the electric current density corresponding to the disk-like source. The energy-momentum tensor  $T_{\alpha\beta}^{\pm}$  in (4a) is taken to be the sum of two distributional components, the purely electromagnetic (trace-free) part and a “material” (trace) part,

$$T_{\alpha\beta}^{\pm} = E_{\alpha\beta}^{\pm} + M_{\alpha\beta}^{\pm}, \quad (5)$$

where  $E_{\alpha\beta}^{\pm}$  is the electromagnetic energy-momentum tensor

$$E_{\alpha\beta} = F_{\alpha\nu} F_{\beta}^{\nu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}, \quad (6)$$

with  $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$  and  $M_{\alpha\beta}^{\pm}$  is an unknown “material” energy-momentum tensor to be obtained. Accordingly, the Einstein-Maxwell equations, in geometrized units such that  $c = 8\pi G = \mu_0 = \epsilon_0 = 1$ , are equivalent to the system of equations

$$G_{\alpha\beta}^{\pm} = R_{\alpha\beta}^{\pm} - \frac{1}{2} g_{\alpha\beta} R^{\pm} = E_{\alpha\beta}^{\pm} + M_{\alpha\beta}^{\pm} \quad (7a)$$

$$H_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} H = Q_{\alpha\beta}, \quad (7b)$$

$$F_{\pm}^{\alpha\beta}{}_{;\beta} = J_{\pm}^{\alpha}, \quad (7c)$$

$$[F^{\alpha\beta}] n_{\beta} = \mathcal{I}^{\alpha}, \quad (7d)$$

where  $H \equiv g^{\alpha\beta} H_{\alpha\beta}$ . The square brackets in expressions such as  $[F^{\alpha\beta}]$  denote the jump of  $F^{\alpha\beta}$  across of the surface  $z = 0$  and  $n_{\beta}$  denotes a unitary vector in the direction normal to it. In the appendix (A) we give the corresponding field equations and the energy-momentum of the halo and of the disk for a sufficiently general metric.

To obtain a solution of the distributional Einstein-Maxwell describing a system composed by a magnetized halo surrounding a thin disk in a conformastationary background, we shall restrict ourselves to the case where the electric potential  $A_t = 0$ . We also conveniently assume the existence of a function  $\phi$  depending only on  $r$  and  $z$  in such a way that the metric (A1) can be written in the form

$$ds^2 = -e^{2\phi} (dt + \omega d\varphi)^2 + e^{-2\beta\phi} (dr^2 + dz^2 + r^2 d\varphi^2), \quad (8)$$

with  $\beta$  an arbitrary constant. Accordingly, for the non-zero components of the energy-momentum tensor of the halo

we have

$$M_{tt}^{\pm} = -e^{2(1+\beta)\phi} \left\{ \beta^2 \nabla \phi \cdot \nabla \phi - 2\beta \nabla^2 \phi + \frac{1}{2} r^{-2} e^{2\beta\phi} \nabla A_{\varphi} \cdot \nabla A_{\varphi} - \frac{3}{4} r^{-2} e^{2(1+\beta)\phi} \nabla \omega \cdot \nabla \omega \right\} \quad (9a)$$

$$M_{t\varphi}^{\pm} = e^{2(1+\beta)\phi} \left\{ \frac{\beta}{2} \nabla \omega \cdot \nabla \phi + \frac{3}{4} r^{-2} e^{2(1+\beta)\phi} \omega \nabla \omega \cdot \nabla \omega - \beta^2 \omega \nabla \phi \cdot \nabla \phi + 2\beta \omega \nabla^2 \phi + \frac{3}{2} \nabla \omega \cdot \nabla \phi - \frac{1}{2} r^{-2} e^{2\beta\phi} \omega \nabla A_{\varphi} \cdot \nabla A_{\varphi} + \frac{1}{2} \nabla^2 \omega - r^{-1} \nabla \omega \cdot \nabla r \right\} \quad (9b)$$

$$M_{rr}^{\pm} = (1-\beta) \nabla^2 \phi - (1-\beta) \phi_{,rr} + (\beta^2 - 2\beta) \phi_{,r}^2 + \phi_{,z}^2 - \frac{1}{2} r^{-2} e^{2\beta\phi} (A_{\varphi,r}^2 - A_{\varphi,z}^2) + \frac{1}{4} r^{-2} e^{2(1+\beta)\phi} (\omega_{,r}^2 - \omega_{,z}^2) \quad (9c)$$

$$M_{rz}^{\pm} = \frac{1}{2} r^{-2} e^{2(1+\beta)\phi} \omega_{,r} \omega_{,z} - (1-\beta^2 + 2\beta) \phi_{,r} \phi_{,z} - (1-\beta) \phi_{,rz} - r^{-2} e^{2\beta\phi} A_{\varphi,r} A_{\varphi,z}, \quad (9d)$$

$$M_{zz}^{\pm} = -\frac{1}{4} r^{-2} e^{2(1+\beta)\phi} (\omega_{,r}^2 - \omega_{,z}^2) + \phi_{,r}^2 - (1-\beta) \phi_{,zz} + (1-\beta) \nabla^2 \phi + (\beta^2 - 2\beta) \phi_{,z}^2 + \frac{1}{2} r^{-2} e^{2\beta\phi} (A_{\varphi,r}^2 - A_{\varphi,z}^2), \quad (9e)$$

$$M_{\varphi\varphi}^{\pm} = r^2 \nabla \phi \cdot \nabla \phi + (1-\beta) r^2 \nabla^2 \phi - (1-\beta) r \nabla \phi \cdot \nabla r - \frac{1}{2} e^{2\beta\phi} \nabla A_{\varphi} \cdot \nabla A_{\varphi} + e^{2(1+\beta)\phi} \left\{ \frac{1}{4} (1 + 3r^{-2} e^{2\beta\phi} \omega^2) \nabla \omega \cdot \nabla \omega - \beta^2 \omega^2 \nabla \phi \cdot \nabla \phi + 2\beta \omega^2 \nabla^2 \phi + \omega \nabla^2 \omega - 2r^{-1} \omega \nabla \omega \cdot \nabla r + (3+\beta) \omega \nabla \omega \cdot \nabla \phi - \frac{1}{2} r^{-2} e^{2\beta\phi} \omega^2 \nabla A_{\varphi} \cdot \nabla A_{\varphi} \right\}. \quad (9f)$$

Whereas the non-zero components of the electric current density on the halo has the form

$$J_{\pm}^t = e^{-(1-3\beta)\phi} \nabla \cdot \{ \omega r^{-2} e^{(1+\beta)\phi} \nabla A_{\varphi} \}, \quad (10a)$$

$$J_{\pm}^{\varphi} = e^{-(1-3\beta)\phi} \nabla \cdot \{ r^{-2} e^{(1+\beta)\phi} \nabla A_{\varphi} \}. \quad (10b)$$

The non-zero components of the surface energy-momentum tensor (SEMT) and the non-zero components of the surface electric current density (SECD) of the disk are given by

$$S_{tt} = 4\beta e^{(2+\beta)\phi} \phi_{,z}, \quad (11a)$$

$$S_{t\varphi} = e^{(2+\beta)\phi} (4\beta \omega \phi_{,z} + \omega_{,z}), \quad (11b)$$

$$S_{rr} = 2(1-\beta) e^{-\beta\phi} \phi_{,z}, \quad (11c)$$

$$S_{\varphi\varphi} = e^{(2+\beta)\phi} \{ (4\beta \omega^2 + 2(1-\beta) r^2 e^{-2(1+\beta)\phi}) \phi_{,z} + 2\omega \omega_{,z} \}, \quad (11d)$$

and

$$\mathcal{J}^t = r^{-2} e^{3\beta\phi} \omega [A_{\varphi,z}], \quad (12a)$$

$$\mathcal{J}^{\varphi} = -r^{-2} e^{3\beta\phi} [A_{\varphi,z}], \quad (12b)$$

respectively. Note that all the quantities are evaluated on the surface of the disk. We will suppose that there is no electric current in the halo, i.e., we assume that

$$J_{\pm}^t = J_{\pm}^{\varphi} = 0. \quad (13)$$

Hence the system of equations (10) is equivalent to the very simple system

$$\nabla \omega \cdot \nabla A_{\varphi} = 0, \quad (14a)$$

$$\nabla \cdot \{ r^{-2} \mathcal{F} \nabla A_{\varphi} \} = 0, \quad (14b)$$

where  $\mathcal{F} \equiv e^{(1+\beta)\phi}$ . If  $\hat{e}_{\varphi}$  is a unit vector in azimuthal direction and  $\lambda$  is an arbitrary function independent of the azimuthal coordinate  $\varphi$ , then one has the identity

$$\nabla \cdot \{ r^{-1} \hat{e}_{\varphi} \times \nabla \lambda \} = 0. \quad (15)$$

The identity (15) may be regarded as the integrability condition for the existence of the function  $\lambda$  defined by

$$r^{-2} \mathcal{F} \nabla A_{\varphi} = r^{-1} \hat{e}_{\varphi} \times \nabla \lambda. \quad (16)$$

Accordingly, the identity (15) implies the equation

$$\nabla \cdot \{\mathcal{F}^{-1} \nabla \lambda\} = 0 \quad (17)$$

for the “auxiliary” potential  $\lambda(r, z)$ . In order to have an explicit form of the metric function  $\phi$  and magnetic potential  $A_\varphi$  we suppose that  $\phi$  and  $A_\varphi$  depend explicitly on  $\lambda$ . Consequently the equation (17) implies

$$-\mathcal{F}^{-1} \mathcal{F}' \nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda = 0, \quad (18)$$

where

$$\mathcal{F}' = (1 + \beta) \mathcal{F} \frac{d\phi}{d\lambda}. \quad (19)$$

Let us assume the very useful simplification

$$\mathcal{F}^{-1} \mathcal{F}' = k, \quad (20)$$

with  $k$  an arbitrary constant. Then, we have  $\mathcal{F} = k_3 e^{k\lambda}$  and

$$-k \nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda = 0, \quad (21)$$

where  $k_3$  is an arbitrary constant. Furthermore, if we assume the existence of a function

$$U = k_4 e^{-k\lambda} + k_5, \quad (22)$$

with  $k_4$  and  $k_5$  arbitrary constants, then

$$\nabla^2 U = -k k_4 e^{-k\lambda} \{-k \nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda\} = 0. \quad (23)$$

Accordingly,  $\lambda$  can be represented in terms of solutions of the Laplace’s equation:

$$e^{k\lambda} = \frac{k_4}{U - k_5}, \quad \nabla^2 U = 0. \quad (24)$$

Hence, the metric potential  $\phi$  can be written in terms of  $U$  as

$$e^{(\beta+1)\phi} = \frac{k_3 k_4}{U - k_5}. \quad (25)$$

To obtain the metric function  $\omega$  we first note that from (16) we have the relationship between  $A_\varphi$  and  $\lambda$ :

$$\nabla A_\varphi = A_{\varphi,r} \hat{e}_r + A_{\varphi,z} \hat{e}_z = r \mathcal{F}^{-1} \hat{e}_\varphi \times (\lambda_{,r} \hat{e}_r + \lambda_{,z} \hat{e}_z). \quad (26)$$

Then we have,  $A_{\varphi,r} = -r \mathcal{F}^{-1} \lambda_{,r}$  and  $A_{\varphi,z} = r \mathcal{F}^{-1} \lambda_{,z}$ , or, in terms of  $U$

$$A_{\varphi,r} = k_6 r U_{,z}, \quad (27a)$$

$$A_{\varphi,z} = -k_6 r U_{,r}, \quad (27b)$$

where  $k_6 = 1/(k k_3 k_4)$ . Furthermore, with (27) into (14a) we arrive to

$$\omega_{,r} U_{,z} - \omega_{,z} U_{,r} = 0, \quad (28)$$

which admits the solution  $\omega = k_\omega U + k_8$ , with  $k_\omega$  and  $k_8$  arbitrary constants. As we know, the line element (8) must reduce to the Minkowski metric at spatial infinity. This means that the gravitational and magnetic fields vanish at large distances from the gravitational source, i.e., it is asymptotically flat. This requires that the constants  $k_3 k_4 = -k_5 = -1$  and  $k_8 = 0$ .

### III. THE SEMT OF THE DISK

In the above section we summarised the procedure to obtain conformastationary axially symmetric relativistic thin disks surrounded by a material halo in presence of a magnetic field. Additionally, we introduced a functional relationship dependence between the metric and the magnetic potential and an harmonic auxiliary function in order to obtain a family of solutions of the distributional Einstein-Maxwell field equations. In short, we used the inverse method, where a solution of the field equations is taken and then the energy-momentum tensor is obtained. Now, the behaviour of the energy-momentum tensor obtained must be investigated in order to find what conditions must be

imposed over the solutions and the parameters that appear in the disk-haloes models in such a way that the energy-momentum tensor can describe a reasonable physical source. When the energy-momentum tensor is diagonal its interpretation is immediate. On the other hand, when the energy-momentum tensor is non diagonal its physical content can be properly analyzed by writing it in the canonical form. Accordingly, to investigate the physical content of the SEMT of the disk we assume that it is possible to express it in the canonical form

$$S_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + Q_\alpha V_\beta + Q_\beta V_\alpha + \Pi_{\alpha\beta}, \quad (29)$$

where  $Q_\alpha V^\alpha = Q^\alpha V_\alpha = 0$ . Consequently, we can say that the disk is constituted by some mass-energy distribution described by the last surface energy-momentum tensor and  $V^\alpha$  is the four velocity of certain observer. Correspondingly,  $\mu$ ,  $P$ ,  $Q_\alpha$  and  $\Pi_{\alpha\beta}$  are then the energy density, the isotropic pressure, the heat flux and the anisotropic tensor on the surface of the disk, respectively. Thus, it is immediate to see that [3]

$$\mu = S_{\alpha\beta} V^\alpha V^\beta, \quad (30a)$$

$$P = \frac{1}{3} \mathcal{H}^{\alpha\beta} S_{\alpha\beta}, \quad (30b)$$

$$Q_\alpha = -\mu V_\alpha - S_{\alpha\beta} V^\beta, \quad (30c)$$

$$\Pi_{\alpha\beta} = \mathcal{H}_\alpha{}^\mu \mathcal{H}_\beta{}^\nu (S_{\mu\nu} - P \mathcal{H}_{\mu\nu}), \quad (30d)$$

where the projection tensor is defined by  $\mathcal{H}_{\mu\nu} \equiv g_{\mu\nu} + V_\mu V_\nu$  and  $\alpha = (t, r, \varphi)$ . It is easy to note that by choosing the angular velocity to be zero in (B7) we have then a fluid comoving in our coordinates system. Hence, we may introduce a suitable reference frame in terms of the local observers tetrad (B3) and (B4) in the form  $\{V^\alpha, I^\alpha, K^\alpha, Y^\alpha\} \equiv \{h_{(t)}^\alpha, h_{(r)}^\alpha, h_{(z)}^\alpha, h_{(\varphi)}^\alpha\}$ , with the corresponding dual tetrad  $\{V_\alpha, I_\alpha, K_\alpha, Y_\alpha\} \equiv \{-h_{(t)\alpha}^{(t)}, h_{(r)\alpha}^{(r)}, h_{(z)\alpha}^{(z)}, h_{(\varphi)\alpha}^{(\varphi)}\}$ . Accordingly, by using (30a), (30b) and (11) we have for the surface energy density and the pressure of the disk

$$\mu = 4\beta e^{\beta\phi} \phi_{,z}, \quad (31)$$

and

$$P = \frac{4}{3}(1 - \beta)e^{\beta\phi} \phi_{,z} = \frac{1 - \beta}{3\beta} \mu, \quad (32)$$

respectively. By introducing (11) into (30c) we obtain for the non-zero components of the heat flux

$$Q_\alpha = -e^{(\beta+1)\phi} \omega_{,z} \delta_\alpha^\varphi. \quad (33)$$

Similarly, by using (30d) and (11) we have for the non-zero components of the anisotropic tensor [4]

$$\Pi_{rr} = \frac{2(1 - \beta)}{3} e^{-\beta\phi} \phi_{,z}, \quad (34a)$$

$$\Pi_{\varphi\varphi} = \frac{2(1 - \beta)}{3} r^2 e^{-\beta\phi} \phi_{,z} = r^2 \Pi_{rr}. \quad (34b)$$

It is important to remark that due to that we used the inverse method, no “a priori” restriction are imposed on the physical properties of the material constituting the disks. The non-zero components of the SEMT of the disks result of “the nature” of the chosen metric and the corresponding solutions. So, in our case, the non-zero component  $S_{rr}$  and  $S_{t\varphi}$  are conditioned by the parameter  $\beta$  and the metric function  $\omega$ . When  $\beta = 1$  the component  $S_{rr}$  vanishes, whereas  $S_{t\varphi} = 0$  when  $\omega$  vanishes. The decomposition (29) was chosen with the aim to describe the SEMT by the more general fluid model. Hence, the heat flux appear here in a “natural” way as a function determined by the metric function  $\omega$  and, consequently, by the “rotation”. Unfortunately, as we can see from (33), this function is oriented along the closed circular orbits and thus its physical interpretation is unclear. It is an issue which remains unanswered in this manuscript, but should be addressed in the future.

Analogously, the SECD of the disk  $\mathcal{J}^\alpha$  can be also written in the canonical form

$$\mathcal{J}^\alpha = \sigma V^\alpha + j Y^\alpha, \quad (35)$$

then  $\sigma$  can be interpreted as the surface electric charge density and  $j$  as the “current of magnetization” of the disk. A direct calculation shows that the surface electric charge density  $\sigma = -V_\alpha \mathcal{J}^\alpha = 0$ , whereas the “current of magnetization” of the disk is given by

$$j = Y_\alpha \mathcal{J}^\alpha = -r^{-1} e^{2\beta\phi} [A_{\varphi,z}], \quad (36)$$

where, as above,  $[A_{\varphi,z}]$  denotes the jump of the  $z$ -derivative of the magnetic potential across of the disk and, all quantities are evaluated on the disk. Thus, by using the results of the precedent section, we can write the surface

energy density, the pressure, the heat flux, the non-zero components of the anisotropic tensor and the current of magnetization on the surface of the disk, respectively, as

$$\mu = \frac{4\beta U_{,z}}{(\beta + 1)(1 - U)^{\frac{2\beta+1}{\beta+1}}} \quad (37a)$$

$$P = \frac{(1 - \beta)}{3\beta} \mu, \quad (37b)$$

$$\mathcal{Q}_\alpha = -\frac{k_\omega U_{,z}}{1 - U} \delta_\alpha^\varphi, \quad (37c)$$

$$\Pi_{rr} = \frac{2(1 - \beta)U_{,z}}{3(1 + \beta)(1 - U)^{\frac{1}{1+\beta}}}, \quad (37d)$$

$$\Pi_{\varphi\varphi} = r^2 \Pi_{rr}, \quad (37e)$$

$$j = -\frac{[U_{,r}]}{k(1 - U)^{\frac{3\beta}{1+\beta}}}, \quad (37f)$$

where, as we know,  $U(r, z)$  is an arbitrary suitable solution of the 2-dimensional Laplace's equation in cylindrical coordinates and  $[U_{,r}]$  denotes of the jump of the  $r$ -derivative of the  $U$  across of the disk. All the quantities are evaluated on the surface of the disk. It is important to note that  $k_\omega$  is a defining constant in (37c). Indeed, when  $k_\omega = 0$  the heat flux  $\mathcal{Q}_\alpha$  vanishes, a feature of the static disk.

#### IV. A PARTICULAR FAMILY OF CONFORMASTATIONARY MAGNETIZED DISK-HALOES SOLUTIONS

In precedent works [1, 5] we have presented a model for a conformastatic relativistic thin disk surrounded by a material electromagnetized halo from the Kuzmin solution of the Laplace's equation in the form

$$U_K = -\frac{m}{\sqrt{r^2 + (|z| + a)^2}}, \quad (a, m > 0). \quad (38)$$

As it is well known,  $\nabla^2 U_K$  must vanish everywhere except on the plane  $z = 0$ . At points with  $z < 0$ ,  $U_K$  is identical to the potential of a point mass  $m$  located at the point  $(r, z) = (0, -a)$ , and when  $z > 0$ ,  $U_K$  coincides with the potential generated by a point mass at  $(0, a)$ . Accordingly, it is clear that  $U_K$  is generated by the surface density of a Newtonian mass

$$\rho_K(r, z = 0) = \frac{am}{2\pi(r^2 + a^2)^{3/2}}. \quad (39)$$

In this work, we present a sort of generalisation of the Kuzmin solution by considering a solution of the Laplace's equation in the form [6],

$$U = -\sum_{n=0}^N \frac{b_n P_n(z/R)}{R^{n+1}}, \quad P_n(z/R) = (-1)^n \frac{R^{n+1}}{n!} \frac{\partial^n}{\partial z^n} \left( \frac{1}{R} \right), \quad (40)$$

$P_n = P_n(z/R)$  being the Legendre polynomials in cylindrical coordinates that was derived in the present form by a direct comparison of the Legendre polynomial expansion of the generating function with a Taylor of  $1/R$  [7].  $R^2 \equiv r^2 + z^2$  and  $b_n$  arbitrary constant coefficients. The corresponding magnetic potential, obtained from (27), is

$$A_\varphi = -\frac{1}{k} \sum_{n=0}^N b_n \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left( \frac{z}{R} \right) \quad (41)$$

where, we have imposed  $A_\varphi(0, z) = 0$  in order to preserve the regularity on the axis of symmetry. Next, to introduce the corresponding discontinuity in the first-order derivatives of the metric potential and the magnetic potential required to define the disk we perform the transformation  $z \rightarrow |z| + a$ . Thus, taking account of (37), the surface energy density of the disk, the heat flux and the non-zero components of the anisotropic tensor are

$$\mu(r) = \frac{4\beta \sum_{n=0}^N b_n (n+1) P_{n+1}(a/R_a) R_a^{-(n+2)}}{(1 + \beta) \left( 1 + \sum_{n=0}^N b_n P_n(a/R_a) R_a^{-(n+1)} \right)^{(2\beta+1)/(\beta+1)}}, \quad (42a)$$

$$Q_\alpha = \frac{k_\omega \delta_\alpha^\varphi \sum_{n=0}^N b_n P_n(a/R_a) R_a^{-(n+1)}}{1 + \sum_{n=0}^N b_n P_n(a/R_a) R_a^{-(n+1)}}, \quad (42b)$$

and

$$\Pi_{rr} = \frac{2(1-\beta) \sum_{n=0}^N b_n(n+1)P_{n+1}(a/R_a)R_a^{-(n+2)}}{3(1+\beta) \left(1 + \sum_{n=0}^N b_n P_n(a/R_a)R_a^{-(n+1)}\right)^{1/(1+\beta)}}, \quad (43)$$

respectively. In the above expressions  $R_a^2 \equiv r^2 + a^2$ . As we know, the other quantities are  $P = (1-\beta)\mu/(3\beta)$  and  $\Pi_{\varphi\varphi} = r^2\Pi_{rr}$ .

The current of magnetization is

$$j = -\frac{2r \sum_{n=0}^N b_n P'_{n+1}(a/R_a)R_a^{-(n+3)}}{k(1 + \sum_{n=0}^N b_n P_n(a/R_a)R_a^{-(n+1)})^{2\beta/(1+\beta)}}, \quad (44)$$

where we have used (37f) and we first assumed that the  $z$ -derivative of the magnetic potential present a finite discontinuity through the disk. In fact, as we have said above, the derivatives of  $U$  and  $A_\varphi$  are continuous functions across of the surface of the disk. We artificially introduce the discontinuity through the transformation  $z \rightarrow |z| + a$ .

In order to illustrate the last solution we consider particular solutions with  $N = 0$  and  $N = 1$ . Then we have  $U_N$  for the two first members of the family of the solutions as follows,

$$U_0 = -\frac{\tilde{b}_0}{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}}, \quad (45a)$$

$$U_1 = -\frac{\tilde{b}_0}{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}} \left\{ 1 + \frac{\tilde{b}_1 (|\tilde{z}| + 1)}{\tilde{b}_0 ((|\tilde{z}| + 1)^2 + \tilde{r}^2)} \right\}, \quad (45b)$$

where  $\tilde{b}_0 = b_0/a$  and  $\tilde{b}_1 = b_1/a^2$  whereas  $\tilde{r} = r/a$  and  $\tilde{z} = z/a$ . For the corresponding magnetic potentials we have then

$$\tilde{A}_{\varphi 0} = -\frac{\tilde{b}_0 (|\tilde{z}| + 1)}{k \sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}}, \quad (46a)$$

$$\tilde{A}_{\varphi 1} = -\frac{\tilde{b}_0 (|\tilde{z}| + 1)}{k \sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}} \left\{ 1 - \frac{\tilde{b}_1 \tilde{r}^2}{\tilde{b}_0 (|\tilde{z}| + 1) ((|\tilde{z}| + 1)^2 + \tilde{r}^2)} \right\}, \quad (46b)$$

where  $\tilde{A}_\varphi = A_\varphi/a$ .

In Fig. 1, we show the dimensionless surface energy densities  $\tilde{\mu}$  as a function of  $\tilde{r}$ . In each case, we plot  $\tilde{\mu}_0(\tilde{r})$  [Fig. 1(a)] and  $\tilde{\mu}_1(\tilde{r})$  [Fig. 1(b)] for different values of the parameter  $\beta$  with  $\tilde{b}_0 = 1$  and  $\tilde{b}_1 = 0.5$ . It can be seen that the surface energy density is everywhere positive fulfilling the energy conditions. It can be observed that for all the values of  $\beta$  the maximum of the surface energy density occurs at the center of the disk and that it vanishes sufficiently fast as  $\tilde{r}$  increases. It can also be observed that the surface energy density in the central region of the disk increases as the values of the parameter  $\beta$  increase. In Fig. 2, we show the dimensionless current of magnetization  $\tilde{j}$  as a function of  $\tilde{r}$ . In each case, we plot  $\tilde{j}_0(\tilde{r})$  [Fig. 2(a)] and  $\tilde{j}_1(\tilde{r})$  [Fig. 2(b)] for different values of the parameter  $\beta$  with  $\tilde{b}_0 = 1$  and  $\tilde{b}_1 = 0.5$ . It can be seen that the current of magnetization is everywhere positive. It can be observed that for all the values of  $\beta$  the current of magnetization is zero at the center of the disk, increases rapidly as one moves away from the disk center, reaches a maximum and later decreases rapidly. It can also be observed that the maximum of the current of magnetization increases as the values of the parameter  $\beta$  decrease. We also computed the functions  $\tilde{\mu}$  and  $\tilde{j}$  for other values of the parameters and, in all the cases, we found the same behaviour. We do not plot the heat flux, it shows a similar behaviour to that of the surface energy density.

Before ending this section we evaluate the constants of motions. From (25) with  $k_3 k_4 = -k_5 = -1$ ,  $k_8 = 0$  and  $k_6 = -1/k$  we have

$$\phi = \frac{1}{1+\beta} \ln \left( \frac{1}{1-U} \right). \quad (47)$$

Then leading term in expansion (40)  $U \approx -b_0/R$  determine ADM mass of the spacetime [8]

$$M_0 = \frac{b_0}{(1+\beta)}, \quad (48)$$

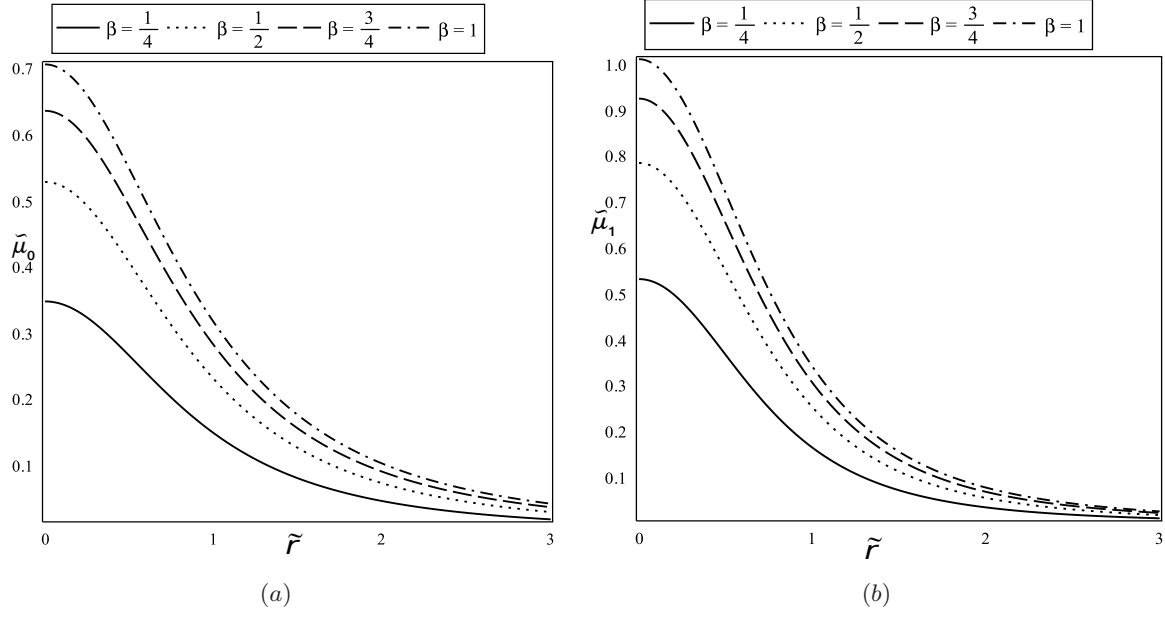


FIG. 1. Dimensionless surface energy densities  $\tilde{\mu}$  as a function of  $\tilde{r}$ . In each case, we plot  $\tilde{\mu}_0(\tilde{r})$  and  $\tilde{\mu}_1(\tilde{r})$  for different values of the parameter  $\beta$  with  $\tilde{b}_0 = 1$  and  $\tilde{b}_1 = 0.5$ .

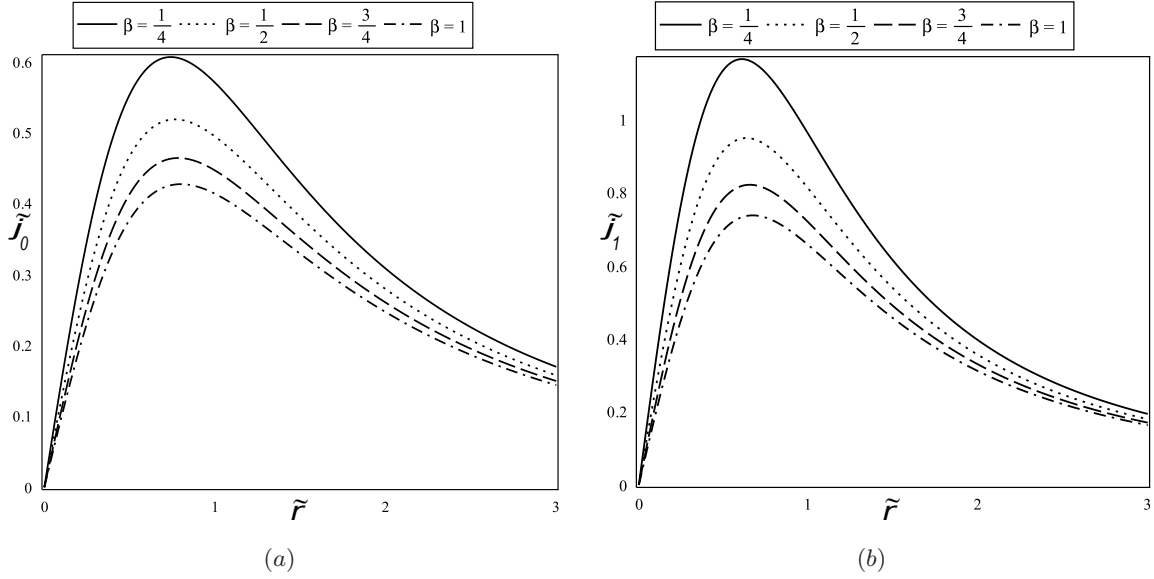


FIG. 2. Dimensionless current of magnetization  $\tilde{\mu}$  as a function of  $\tilde{r}$ . In each case, we plot  $\tilde{\mu}_0(\tilde{r})$  and  $\tilde{\mu}_1(\tilde{r})$  for different values of the parameter  $\beta$  with  $\tilde{b}_0 = 1$  and  $\tilde{b}_1 = 0.5$ .

its total angular momentum

$$L_{M0} = \frac{1}{2} k_{\omega} b_0 \quad (49)$$

and the total magnetic dipole moment

$$L_{B0} = \frac{b_0}{k}. \quad (50)$$



We thus see that constants  $k$  and  $k_\omega$  defines the gyromagnetic ratio  $L_{M0}/L_{B0} = (kk_\omega)/2$ .

## V. CONCLUDING REMARKS

We have used the formalism presented in [1] to obtain an exact relativistic model describing a system composed of a thin disk surrounded by a magnetized halo in a conformastationary space-time background. The model was obtained by solving the Einstein-Maxwell distributional field equations through the introduction of an auxiliary harmonic function that determines the functional dependence of the metric components and the electromagnetic potential under the assumption that the energy-momentum tensor can be expressed as the sum of two distributional contributions, one due to the electromagnetic part and the other due to a material part. As we can see, due to that the spacetime here considered is non-static (conformastationary) this distributional approach allows us a strongly non-linear partial equation system. We have considered for simplicity the astrophysical consistent case in that there is not electric charge on the halo region. Consequently, it has been obtained that the charge density on the disk region is zero.

To analyse the physical content of the energy-momentum tensor of the disks we expressed it in the canonical form and we projected it in a comoving frame defined trough of the local observers tetrad. This analysis has allowed us to give a complete dynamical description of the system in terms of two parameters (i.e  $\beta$  and  $k_\omega$ ) which determine the matter content of the disks. So, in this paper we presented for first time the complete analysis of the most general energy-momentum tensor of the disks that can be obtained from conformastationary axially symmetric solutions of the Einstein-Maxwell equations.

The expressions obtained here are the generalisations of the corresponding expressions for the conformastatic disks without isotropic pressure, stress tensor or heat flow presented in [1]. Indeed, when the parameter  $\beta$  in the metric is equal to one the isotropic pressure and the anisotropic tensor on the material constituting the disks disappear. In a similar way, when the parameter  $k_\omega$  is equal to zero the heat flux on the disk vanishes, a feature of the static systems. Furthermore, when we take simultaneously  $k_\omega = 0$  and  $\beta = 1$ , the results here presented describe the energy-momentum tensor the disks presented in [1] for the special case when the electric potential vanishes. Moreover, to illustrate the application of the formalism we have considered specific solutions in which the gravitational and magnetic potential are completely determined by a “generalization” of the Kuzmin solution of the Laplace’s equation. Accordingly, we have obtained conformastationary magnetized thin disks of infinite radius, generated from a Newtonian gravitational potential of a static axisymmetric distribution of matter. Hence, when a particular value of the parameters  $b_0, b_1, \beta, k_\omega$  is taken, the conformastatic disks without radial pressure presented in [1] are obtained.

Since all the relevant quantities show a physically reasonable behaviour, we conclude that the solution presented here can be useful to describe the gravitational and electromagnetic field of a conformastationary thin disk surrounded by a halo in the presence of an electromagnetic field. In a subsequent work we will present a detailed analysis of the energy-momentum tensor of the halo as well as a thermodynamic analysis of the disk-halo.

## ACKNOWLEDGEMENT

The author would like to thank the anonymous reviewer for their valuable comments and suggestions to improve the quality of the paper. The author also wishes to acknowledge useful discussions with C. S. Lopez-Monsalvo and H. Quevedo.

## Appendix A: The Einstein-Maxwell equations and the thin-disk-halo system

Inspired by inverse method techniques, let us assume that the solution has the general form [6, 9]

$$ds^2 = -f^2(dt + \omega d\varphi)^2 + \Lambda^4[dr^2 + dz^2 + r^2 d\varphi^2], \quad (A1)$$

where we have introduced the cylindrical coordinates  $x^\alpha = (t, r, z, \varphi)$  in which the metric function  $f, \Lambda$  and  $\omega$  and the electromagnetic potential,  $A_\alpha = (A_t, 0, 0, A_\varphi)$  depend only on  $r$  and  $z$ . Accordingly, the non-zero components of the

energy-momentum tensor of the halo,  $M_{\alpha\beta}^{\pm} = G_{\alpha\beta}^{\pm} - E_{\alpha\beta}^{\pm}$ , are given by

$$M_{tt}^{\pm} = \frac{1}{4r^2\Lambda^8} \{3f^4\nabla\omega \cdot \nabla\omega - 16r^2f^2\Lambda^3\nabla^2\Lambda - 2f^2\nabla A_{\varphi} \cdot \nabla A_{\varphi} + 4f^2\omega\nabla A_t \cdot \nabla A_{\varphi} - 2(r^2\Lambda^4 + f^2\omega^2)\nabla A_t \cdot \nabla A_t\}, \quad (\text{A2a})$$

$$M_{t\varphi}^{\pm} = \frac{1}{4r^2\Lambda^8} \{-4r^2f^2\Lambda^3\nabla\omega \cdot \nabla\Lambda + 3f^4\omega\nabla\omega \cdot \nabla\omega - 16r^2f^2\Lambda^3\omega\nabla^2\Lambda + 6r^2\Lambda^4f\nabla\omega \cdot \nabla f + 2\Lambda^4r^2f^2\nabla^2\omega - 4r^2f^2\Lambda^4\nabla\omega \cdot \nabla r - 2\omega f^2\nabla A_{\varphi} \cdot \nabla A_{\varphi} + 2(\omega\Lambda^4r^2 - f^2\omega^3)\nabla A_t \cdot \nabla A_t + 4(f^2\omega^2 - r^2\Lambda^4\nabla A_t \cdot \nabla A_{\varphi})\} = M_{\varphi t}^{\pm}, \quad (\text{A2b})$$

$$M_{rr}^{\pm} = -\frac{1}{4r^2\Lambda^4f^2} \{-f^4(\omega_r^2 - \omega_z^2) - 4\Lambda^4r^2f\nabla^2f + 4r^2f\Lambda^4f_{rr} - 8r^2f^2\Lambda^3\nabla^2\Lambda + 8r^2f^2\Lambda^3\Lambda_{rr} - 16r^2\Lambda^3ff_{,r}\Lambda_{,r} + 8f^2\Lambda^2r^2\Lambda_{,z}^2 - 16f^2r^2\Lambda^2\Lambda_{,r}^2 + 2f^2(A_{\varphi,r}^2 - A_{\varphi,z}^2) - 2(\Lambda^4r^2 - f^2\omega^2)(A_{t,r}^2 - A_{t,z}^2) - 4\omega f^2(A_{t,r}A_{\varphi,r} - A_{t,z}A_{\varphi,z})\}, \quad (\text{A2c})$$

$$M_{rz}^{\pm} = \frac{1}{2r^2\Lambda^4f^2} \{4r^2f\Lambda^3(\Lambda_{,r}f_{,z} + \Lambda_{,z}f_{,r}) + 12r^2f^2\Lambda^2\Lambda_{,r}\Lambda_{,z} + f^4\omega_{,r}\omega_{,z} - 4r^2f^2\Lambda^3\Lambda_{,rz} - 2r^2\Lambda^4ff_{,rz} - 2(f^2\omega^2 - \Lambda^4r^2)A_{t,r}A_{t,z} + 2f^2\omega(A_{\varphi,r}A_{t,z} + A_{t,r}A_{\varphi,z}) - 2f^2A_{\varphi,r}A_{\varphi,z}\}, \quad (\text{A2d})$$

$$M_{zz}^{\pm} = \frac{1}{4r^2\Lambda^4f^2} \{-f^4(\omega_r^2 - \omega_z^2) + 4r^2\Lambda^4f\nabla^2f - 4r^2\Lambda^4ff_{,zz} + 8r^2f^2\Lambda^3\nabla^2\Lambda - 8r^2f^2\Lambda^3\Lambda_{,zz} - 8r^2f^2\Lambda^2\Lambda_{,r}^2 + 16r^2\Lambda^3ff_{,z}\Lambda_{,z} + 16r^2f^2\Lambda^2\Lambda_{,z}^2 + 2f^2(A_{\varphi,r}^2 - A_{\varphi,z}^2) - 4\omega f^2(A_{t,r}A_{\varphi,r} - A_{t,z}A_{\varphi,z}) - 2(\Lambda^4r^2 - f^2\omega^2)(A_{t,r}^2 - A_{t,z}^2)\}, \quad (\text{A2e})$$

$$M_{\varphi\varphi}^{\pm} = \frac{1}{4r^2\Lambda^8f^2} \{4r^4f\Lambda^8\nabla^2f - 4r^3\Lambda^8f\nabla f \cdot \nabla r + f^4(\Lambda^4r^2 + 3\omega^2f^2)\nabla\omega \cdot \nabla\omega - 16r^2f^4\omega^2\Lambda^3\nabla^2\Lambda + 4r^2\Lambda^4f^4\omega\nabla^2\omega - 8r^4f^4\omega\nabla\omega \cdot \nabla r + 12\Lambda^4r^2\omega f^3\nabla\omega \cdot \nabla f - 8r^2\omega f^4\Lambda^3\nabla\omega \cdot \nabla\Lambda + 8f^2\Lambda^7r^4\nabla^2\Lambda - 8f^2\Lambda^7r^3\nabla\Lambda \cdot \nabla r - 8r^4f^2\Lambda^6\nabla\Lambda \cdot \nabla\Lambda - 2(f^2\Lambda^4r^2 + f^4\omega^2)\nabla A_{\varphi} \cdot \nabla A_{\varphi} - 4(f^2\omega\Lambda^4r^2 - \omega^3f^4)\nabla A_{\varphi} \cdot \nabla A_t - 2(\Lambda^8r^4 - 2\Lambda^4r^2f^2\omega^2 + f^4\omega^4)\nabla A_t \cdot \nabla A_t\}. \quad (\text{A2f})$$

Furthermore, the electric current density of the halo reads

$$J_{\pm}^t = \frac{1}{\Lambda^6f} \nabla \cdot \{\Lambda^2f^{-1}\nabla A_t + r^{-2}\omega f\Lambda^{-2}(\nabla A_{\varphi} - \omega\nabla A_t)\}, \quad (\text{A3a})$$

$$J_{\pm}^{\varphi} = \frac{1}{\Lambda^6f} \nabla \cdot \{r^{-2}f\Lambda^{-2}(\nabla A_{\varphi} - \omega\nabla A_t)\}. \quad (\text{A3b})$$

The discontinuity in the  $z$ -direction of  $Q_{\alpha\beta}$  and  $\mathcal{I}^{\alpha}$  defines, respectively, the surface energy-momentum tensor (SEMT) and the surface electric current density (SECD) of the disk  $S_{\alpha\beta}$ , more precisely

$$S_{\alpha\beta} \equiv \int Q_{\alpha\beta}\delta(z)ds_n = \sqrt{g_{zz}}Q_{\alpha\beta}, \quad (\text{A4a})$$

$$\mathcal{J}^{\alpha} \equiv \int \mathcal{I}^{\alpha}\delta(z)ds_n = \sqrt{g_{zz}}\mathcal{I}^{\alpha} \quad (\text{A4b})$$

where  $ds_n = \sqrt{g_{zz}}dz$  is the “physical measure” of length in the direction normal to the  $z = 0$  surface. Accordingly, the non-zero components of the SEMT for the line element (A1) are given by

$$S_t^t = \frac{1}{r^2\Lambda^6} \{-f^2\omega\omega_{,z} + 8r^2\Lambda^3\Lambda_{,z}\} \quad (\text{A5a})$$

$$S_{\varphi}^t = -\frac{1}{r^2\Lambda^6f} \{-4r^2\omega f\Lambda^3\Lambda_{,z} + \omega^2f^3\omega_{,z} + 2r^2\Lambda^4\omega f_{,z} + r^2f\Lambda^4\omega_{,z}\}, \quad (\text{A5b})$$

$$S_r^r = \frac{2}{\Lambda^3f} \{2f\Lambda_{,z} + \Lambda f_{,z}\}, \quad (\text{A5c})$$

$$S_t^{\varphi} = \frac{f^2}{r^2\Lambda^6}\omega_{,z} \quad (\text{A5d})$$

$$S_{\varphi}^{\varphi} = \frac{1}{r^2\Lambda^6f} \{2r^2\Lambda^4f_{,z} + 4r^2\Lambda^3f\Lambda_{,z} + \omega f^3\omega_{,z}\}, \quad (\text{A5e})$$

whereas the non-zero components of the (SECD) are

$$\mathcal{J}^t = \frac{1}{r^2 \Lambda^6 f^2} \{ \omega f^2 [A_{\varphi, z}] + (r^2 \Lambda^4 - f^2 \omega^2) [A_{t, z}] \}, \quad (\text{A6a})$$

$$\mathcal{J}^\varphi = \frac{1}{r^2 \Lambda^6} \{ \omega [A_{t, z}] - [A_{\varphi, z}] \}, \quad (\text{A6b})$$

where all the quantities are evaluated on the surface of the disk.

## Appendix B: The local observers

We write the metric (8) in the form

$$ds^2 = -F(dt + \omega d\varphi)^2 + F^{-\beta} [dr^2 + dz^2 + r^2 d\varphi^2], \quad (\text{B1})$$

where we have rewritten  $F = e^{2\phi}$ . The tetrad of the local observers  $h^{(\alpha)}_\mu$ , in which the metric has locally the form of Minkowskian metric

$$ds^2 = \eta_{(\mu)(\nu)} \mathbf{h}^{(\mu)} \otimes \mathbf{h}^{(\nu)}, \quad (\text{B2})$$

is given by

$$h^{(t)}_\alpha = F^{1/2} \{1, 0, 0, \omega\}, \quad (\text{B3a})$$

$$h^{(r)}_\alpha = F^{-\beta/2} \{0, 1, 0, 0\}, \quad (\text{B3b})$$

$$h^{(z)}_\alpha = F^{-\beta/2} \{0, 0, 1, 0\}, \quad (\text{B3c})$$

$$h^{(\varphi)}_\alpha = F^{-\beta/2} \{0, 0, 0, r\}. \quad (\text{B3d})$$

$$(\text{B3e})$$

The dual tetrad reads

$$h_{(t)}^\alpha = F^{-1/2} \{1, 0, 0, 0\}, \quad (\text{B4a})$$

$$h_{(r)}^\alpha = F^{\beta/2} \{0, 1, 0, 0\}, \quad (\text{B4b})$$

$$h_{(z)}^\alpha = F^{\beta/2} \{0, 0, 1, 0\}, \quad (\text{B4c})$$

$$h_{(\varphi)}^\alpha = \frac{F^{\beta/2}}{r} \{-\omega, 0, 0, 1\}. \quad (\text{B4d})$$

$$(\text{B4e})$$

The circular velocity of the system disk-halo can be modelled by a fluid space-time whose circular velocity  $V^\alpha$  can be written in terms of two Killing vectors  $t^\alpha$  and  $\varphi^\alpha$ ,

$$V^\alpha = V^t (t^\alpha + \Omega \varphi^\alpha), \quad (\text{B5})$$

where

$$\Omega \equiv \frac{u^\varphi}{u^t} = \frac{d\varphi}{dt} \quad (\text{B6})$$

is the angular velocity of the fluid as seen by an observer at rest at infinity. The velocity satisfy the normalization  $V_\alpha V^\alpha = -1$ . Accordingly for the metric (B1) we have

$$(V^t)^2 = \frac{1}{-t^\alpha t_\alpha - 2\Omega t^\alpha \varphi_\alpha - \Omega \varphi^\alpha \varphi_\alpha}, \quad (\text{B7})$$

with

$$t^\alpha t_\alpha = g_{tt} = -F \quad (\text{B8a})$$

$$t^\alpha \varphi_\alpha = g_{t\varphi} = -\omega F \quad (\text{B8b})$$

$$\varphi^\alpha \varphi_\alpha = g_{\varphi\varphi} = r^2 F^{-\beta} (1 - F^{1+\beta} \frac{\omega^2}{r^2}). \quad (\text{B8c})$$

Consequently, we write the velocity as

$$V^t = \frac{1}{F^{1/2}(1 + \omega\Omega)\sqrt{1 - V_{Loc}^2}}, \quad (\text{B9})$$

where

$$V_{Loc} \equiv \frac{r\Omega}{F^{(1+\beta)/2}(1 + \omega\Omega)}, \quad (\text{B10})$$

is the velocity as measured by the local observers.

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